Normal subgroups and quotient groups

(story so far)

- If G and K are groups and $\phi : G \to K$ is a homomorphism, then $\ker(\phi) \trianglelefteq G$. (important fact 2 from Normal subgroups)

- Suppose G is a group and $H \leq G$. The rule (previous video)

$$(g_1H)(g_2H) = (g_1g_2)H, \quad \forall g_1H, g_2H \in G/H,$$

is a well-defined binary operation on G/H if and only if $H \trianglelefteq G$.

Theorem: If G is a group then a subgroup $H \leq G$ is normal if and only if it is the kernel of a homomorphism $\phi : G \to K$, for some group K.

Pf: $\Leftarrow$ has already been established.

$\Rightarrow$: Suppose $H \trianglelefteq G$ and let $K = G/H$. Define $\phi : G \to K$ by $\phi(g) = gH$. Then:

- ϕ is a hom. ✓

$$\forall g_1, g_2 \in G, \quad \phi(g_1g_2) = (g_1g_2)H = (g_1H)(g_2H) = \phi(g_1)\phi(g_2).$$

- $\ker(\phi) = H$. ✓

$$\ker(\phi) = \{g \in G : \phi(g) = eH\} = \{g \in G : g \in H\} = H. \quad \blacksquare$$

First isomorphism theorem:

If $\phi: G \to K$ is a homomorphism of groups then $\ker(\phi) \trianglelefteq G$ and

$$G/\ker(\phi) \cong \phi(G).$$

Pf: Write $H = \ker(\phi)$. We already know that $H \trianglelefteq G$ and $\phi(G) \leq K$.

Define $\tau: G/H \to \phi(G)$ by $\tau(gH) = \phi(g)$. This map is:

- Well-defined ✓

  If $gH = g'H$ then $g' = gh$ for some $h \in H$

  $$\Rightarrow \tau(gH) = \phi(g) = \phi(g)\overset{=e}{\phi(h)} \underset{\substack{\uparrow \\ \phi \text{ is a hom.}}}{=} \phi(gh) = \phi(g') = \tau(g'H).$$

- a homomorphism ✓

  $\forall gH, g'H \in G/H$,

  $$\tau((gH)(g'H)) = \tau((gg')H) = \phi(gg') = \phi(g)\phi(g') = \tau(gH)\tau(g'H).$$

- injective ✓

  If $gH, g'H \in G/H$ and $\tau(gH) = \tau(g'H)$, then

  $$\phi(g) = \phi(g') \Rightarrow e_G = \phi(g)^{-1}\phi(g') = \phi(g^{-1}g')$$

  $$\Rightarrow g^{-1}g' \in H \Rightarrow g^{-1}g' = h \text{ for some } h \in H$$

  $$\Rightarrow g' = gh \Rightarrow g' \in gH \Rightarrow g'H = gH.$$

• surjective: ✓

$\forall k \in \phi(G)$, $k = \phi(g)$ for some $g \in G \Rightarrow k = \tau(gH)$.

Conclusion: $G/H \cong \tau(G) = \phi(G)$. ▨

Examples:

1) The map $\phi : GL_2(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$, $\phi(A) = \det(A)$, is a homomorphism with

(see "Basic properties of homomorphisms")

$$\ker(\phi) = \{A \in GL_2(\mathbb{R}) : \det(A) = 1\} = SL_2(\mathbb{R})$$

Therefore $SL_2(\mathbb{R}) \trianglelefteq GL_2(\mathbb{R})$,

(ϕ is surjective)

and $$GL_2(\mathbb{R}) / SL_2(\mathbb{R}) \cong \phi(GL_2(\mathbb{R})) = \mathbb{R} \setminus \{0\}.$$

2) $G = D_{16} = \langle r, s \mid r^8 = s^2 = e,\ rs = sr^{-1} \rangle$

$K = C_4 \times C_4 = \langle x \rangle \times \langle y \rangle = \{(x^i, y^j) : 0 \le i, j \le 3\}$

Let $\phi : \{r, s\} \to K$ be defined by $\phi(r) = (x^2, e)$, $\phi(s) = (e, y^2)$.

- ϕ extends to a hom. $\phi : G \to K$

$$\phi(r)^8 = (x^2, e)^8 = (e, e)$$

$$\phi(s)^2 = (e, y^2)^2 = (e, y^4) = (e, e)$$

K is Abelian

$$\phi(r)\phi(s) = (x^2, e)(e, y^2) = (e, y^2)(x^2, e) = \phi(s)\phi(r)^{-1}$$

$(x^2, e) = (x^2, e)^{-1}$

- $\phi(D_8) = \langle (x^2,e),(e,y^2)\rangle \cong V_4$

All non-identity elements in $\langle (x^2,e),(e,y^2)\rangle$ have order 2.

- $\ker(\phi) = \langle r^2\rangle = \{e, r^2, r^4, r^6\}$

For $0 \le i \le 7$, $\phi(r^i) = \phi(r)^i = (x^2,e)^i = (x^{2i},e) = (e,e) \Leftrightarrow i = 0, 2, 4,$ or 6.

and $\phi(sr^i) = \phi(s)\phi(r)^i = (e,y^2)(x^2,e)^i = (x^{2i},y^2) \neq (e,e)$.

Therefore, by the 1st isom. thm., $\langle r^2\rangle \trianglelefteq D_{16}$ and

$$D_{16}/\langle r^2\rangle \cong V_4$$

3) Let G be any Abelian group, and consider the map

$\phi: G\times G \to G$ defined by $\phi(g,h) = gh^{-1}$.

- ϕ is a homomorphism

$\forall (g_1,h_1),(g_2,h_2) \in G\times G$,

$$\phi((g_1,h_1)(g_2,h_2)) = \phi((g_1g_2, h_1h_2)) = g_1g_2(h_1h_2)^{-1}$$

G is Abelian

$$= g_1g_2h_2^{-1}h_1^{-1} = g_1h_1^{-1}g_2h_2^{-1} = \phi((g_1,h_1))\phi((g_2,h_2))$$

- ϕ is surjective

$\forall g \in G$, $\phi((g,e)) = ge^{-1} = g$.

- $H = \ker(\phi) = \{(g,g) : g \in G\}$

$(g,h) \in H \Leftrightarrow gh^{-1} = e \Leftrightarrow g = h$.

Therefore by the 1st isom. thm., $(G\times G)/H \cong G$.